Complex Analysis: Midterm Exam

Aletta Jacobshal 01, Monday 18 December 2017, 09:00–11:00
Exam duration: 2 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of each of your exam sheets and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you **must** explain why the conditions for using such results are satisfied.
- 10 points are "free". There are 4 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (20 points)

Consider the function

$$f(z) = \frac{\bar{z}}{1-z}.$$

(a) (8 points) Write $f(z)$ in the form $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$.

(b) (12 points) Use the Cauchy-Riemann equations to determine where $f(z)$ is differentiable.

Question 2 (20 points)

The principal value of arcsin is defined as

$$\operatorname{Arcsin}(z) = -i \operatorname{Log}\left(iz + \sqrt{1-z^2}\right),$$

where $\sqrt{z}$ denotes the principal value of $z^{1/2}$ (consider known that: for $x > 0$, $\sqrt{x}$ equals the real square root; for $x < 0$, $\sqrt{x} = i\sqrt{|x|}$; and that $\sqrt{0} = 0$).

(a) (10 points) Compute $\operatorname{Arcsin}(1)$, $\operatorname{Arcsin}(i)$, and $\operatorname{Arcsin}(2)$.

(b) (10 points) Show that the half-line on the complex plane defined by $z \in \mathbb{R}$ with $z > 1$ is a branch cut of Arcsin.

Question 3 (20 points)

Consider the closed unit disk $U = \{z \in \mathbb{C} : |z| \leq 1\}$. Show that

$$\max_{z \in U} |az^n + b| = |a| + |b|.$$

Here $a, b \in \mathbb{C}$ are constant and n is an integer with $n \geq 1$.

Question 4 (30 points)

(a) (15 points) Compute the value of the integral

$$\int_\Gamma \frac{e^z}{(z+1)(z^2-4)}\,dz,$$

where Γ is the closed contour shown in Figure 1.

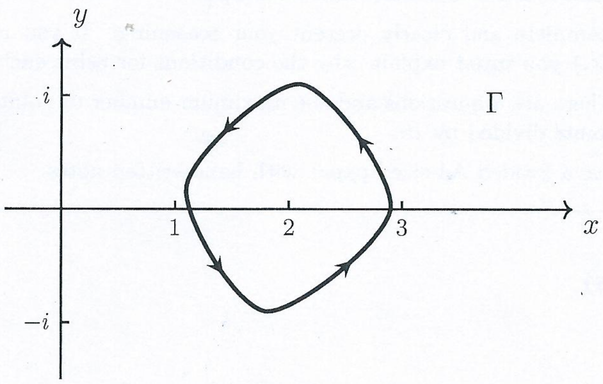

Figure 1: Contour Γ for Question 4(a).

(b) (15 points) Compute the value of the integral

$$\int_C (\bar{z} + z^2 \sin z)\,dz,$$

where C is the circle $|z-1| = 1$ traversed in the clockwise direction.